EN5101 Digital Control Systems
Lyapunov Stability

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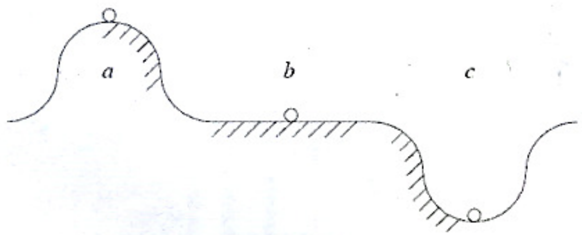
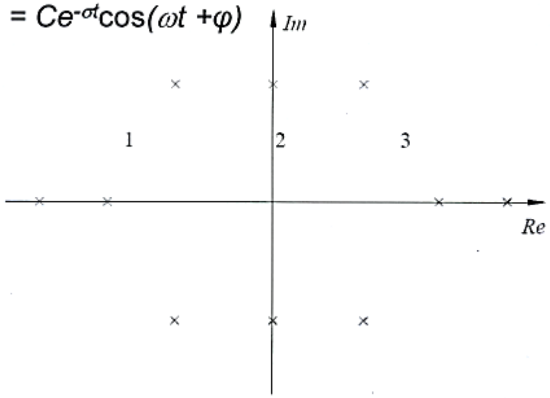
Stability Definition

- A system is stable if the output is bounded for all bounded inputs. Stability is property of a system, independent of input signal.
- Equilibrium states can be unstable equilibrium (point *a*), *neutral equilibrium (region b), or stable* equilibrium (point *c*); *this is demonstrated in the diagram*



Observer, Plant, and Control Law

- Simple test for system stability: The real part of all poles must be negative. Poles are eigenvalues of system dynamics matrix **A**
- Characteristic equation $|sI - A| = 0$
- *Poles, eigen values:* $s_i = \sigma_i \pm j\omega_i$
- Transient solution $y(t) = Ce^{-\sigma t}\cos(\omega t + \varphi)$



Classical Methods

1. If all $\sigma_i < 0$ *stable*
2. If any $\sigma_i = 0$ *marginally stable (assuming the remaining* σ_i *are negative)*
3. If any $\sigma_i > 0$ *unstable*

- **Routh-Hurwitz criterion -** determine stability based on transfer function coefficients without actually calculating poles.
- **Root-locus method -** graphical method to vary feedback gain *k to determine ranges for stability* and control transient response
- **Nyquist Method –** OLTF frequency response (graphical)

Stability from Energy Point of View

- Applicable to linear/nonlinear as well as static/dynamic systems
  - If dE/dt < 0 then stable
  - If dE/dt < 0 then marginally stable
  - If dE/dt > 0 then unstable

Lyapunov Background

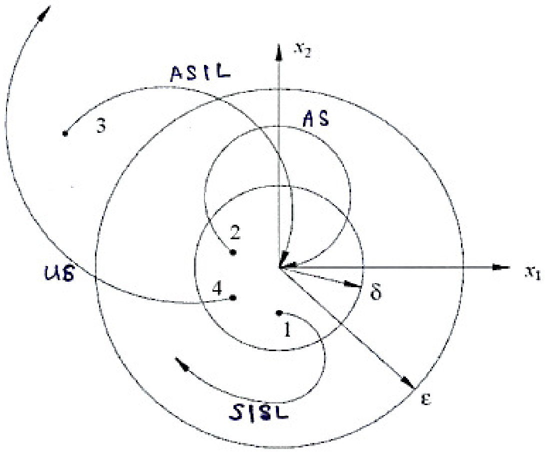
- Any time varying nonlinear system can be represented as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$
$$y = \mathbf{C}\mathbf{x}$$

neglect forcing input *u*

- A state is an equilibrium state $\mathbf{x}_e$ if $\mathbf{f}(\mathbf{x}_e, t) = 0$ for all *t*.
- For linear time invariant systems, $d\mathbf{x}/dt = \mathbf{f}(\mathbf{X}, t) = \mathbf{A}\mathbf{x}$ and there is one unique equilibrium state $\mathbf{x}_e$ if **A** is *nonsingular*. There can be infinitely many equilibrium states $\mathbf{x}_e$ if **A** is *singular*
- We can always shift an equilibrium state $\mathbf{x}_e$ to zero by coordinate shifts: f(0,*t*) = *0 for all t*.

- Consoder a hyperspherical region of radius *k* about an equilibrium state $\mathbf{x}_e$ (using Euclidean norm)
- $\|\mathbf{x} - \mathbf{x}_e\| = k$ where $\|\mathbf{x} - \mathbf{x}_e\| = \sqrt{(x - x_{1e})^2 + (x - x_{2e})^2 + \cdots + (x - x_{ne})^2}$
- Define two such spherical regions $\|\mathbf{x} - \mathbf{x}_e\| = \delta$ and $\|\mathbf{x} - \mathbf{x}_e\| = \varepsilon$ with $\delta < \varepsilon$



1. An equilibrium state $\mathbf{x}_e$ is said to be ***stable in the sense of Lyapunov*** (stability I.S.L.) if trajectories starting within δ do not leave the ε region as *t* increases indefinitely

2. An equilibrium state $\mathbf{x}_e$ is said to be ***asymptotically stable*** if trajectories starting within δ converge to $\mathbf{x}_e$ without leaving the ε region as *t* increases indefinitely. [This case is preferable to stability I.S.L.]

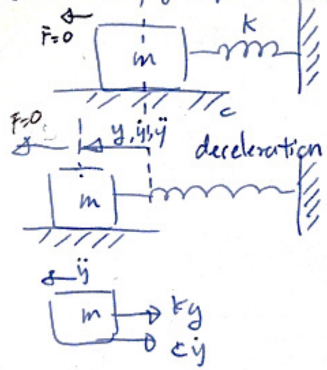
3. An equilibrium state $\mathbf{x}_e$ is said to be ***asymptotically stable in the large*** *if trajectories starting from anywhere in the hyperspace* converge to $\mathbf{x}_e$. There must be only one equilibrium state in the whole state space

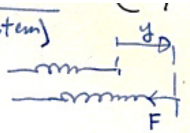
4. An equilibrium state $\mathbf{x}_e$ is said to be ***unstable*** if trajectories starting within δ leaves the ε region as t increases

Lyapunov 1st Method (Indirect, Intuitive)

- For any system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t)$
- No need of solving state space model
- Non-unique, intuitive method

(Neglect forcing input study the system)





$F = ky$ K-stiffness

incremental stored energy in the spring

$\delta E = F\,dy$

$\therefore E_{spring} = \int F\,dy$

$= \int ky\,dy$

Class Exercise: Show that the System is stable using energy

Force balance, $m\ddot{y} = F - ky - c\dot{y}$

$m\ddot{y} = -(ky + c\dot{y})$ —①

$E_{spring} = \frac{1}{2}ky^2$ —②

Kinetic Energy of the mass

$E_K = \frac{1}{2}m\dot{y}^2$ —③

Total Energy of the System

$E_{total} = E_{spring} + E_{kinetic}$ (1) (2)

$E = \frac{1}{2}ky^2 + \frac{1}{2}m\dot{y}^2$

$\frac{dE}{dt} = ky\dot{y} + m\dot{y}\ddot{y}$

$= (ky + m\ddot{y})\dot{y}$ —④

From ① $\Rightarrow$ $m\ddot{y} + ky = -c\dot{y}$ put on ④

$\frac{dE}{dt} = -c\dot{y}\dot{y}$

$= -c(\dot{y})^2$

$< 0 \quad \forall\ c > 0$ (positive damping)

System is stable for $c > 0$

Assignment: Check stability of the non linear plant

$\dot{x} = f(x,t)$ where

$\dot{x}_1 = -x_1 + x_2 + x_1(x_1^2 + x_2^2)$ —(1)

$\dot{x}_2 = -x_1 - x_2 + x_2(x_1^2 + x_2^2)$ —(2)

Scalar Functions of State Vector

- Scalar function of state v(x) is +ve definite if $V(\mathbf{x}) > 0 \ \ \forall x \in \Omega$ and $V(\mathbf{x})=0$ for $\mathbf{x}=0$
- Scalar function of state v(x) is -ve definite if $V(\mathbf{x}) < 0 \ \ \forall x \in \Omega$ and $V(\mathbf{x})=0$ for $\mathbf{x}=0$

$\mathbf{V}_1(\mathbf{x}) = x_1^2 + x_2^2$ is + ve definite $\quad x^T I x = (x_1, x_2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

$\mathbf{V}_2(\mathbf{x}) = (x_1 + x_2)^2$ is + ve semi - definite

$\mathbf{V}_3(\mathbf{x}) = x_1^2 + x_1 x_2$ is indefinite

$\mathbf{V}_4(\mathbf{x}) = -x_1^2 - (x_1 + x_2)^2$ is - ve definite

Lyapunov 2^{nd} Method (Direct)

- If a +ve definite function V(**x**) can be found such that dV(**x**)/dt is –ve definite, this equilibrium state is asymptotically stable
- Qudratic form of V(**x**)=$\mathbf{x}^T\mathbf{P}\mathbf{x}$; **P** is real symmetric and +ve definite

- Sylvester's Criterion – **P** is +ve definite if all principal minors are positive. Principal minors are submatrix determinants starting with scalar p11 and proceeding (with p11 included as the first term in each) until the determinant of the entire **P**
- **P** is +ve semi-definite if all principal minors are non-negative (at least one zero)
- Eg:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 2 \\ 1.5 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

pm1 pm2 pm3

In MatLab
P=[1 0 2;1.5 0.5 2;1 2 3]
submat1=P(1,1)
submat2=P(1:2,1:2)
submat3=P(1:3,1:3)
pm1=det(submat1)
pm2=det(submat2)
pm3=det(submat3)

Answer
pm1=1.0
pm2=0.5
pm3=2.5

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$$

$$\dot V(\mathbf{x}) = \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\dot{\mathbf{P}}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} = \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}}$$

for constant **A** and constant **P**

$$\dot V(\mathbf{x}) = (\mathbf{A}\mathbf{x})^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}(\mathbf{A}\mathbf{x}); \text{as } \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\dot V(\mathbf{x}) = \mathbf{x}^T\mathbf{A}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{A}\mathbf{x} = \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x}$$

If $\dot V(\mathbf{x})$ is to be -ve definite $(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})$ has to be -ve definite

Lets introduce any known +ve definite **Q** such that

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$$

Determine symmetric P and check for +ve definiteness
(easy way)

Example (2^{nd} method)

Determine the stability condition of $\dot{x} = Ax + Bu$ where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

The stability condition is

$A^TP + PA = -Q$ lets set $Q = I_2$ +ve def^{te}

solve for symmetric $P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and verify its +ve definiteness

$$\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2b & -2c \\ a-3b & b-3c \end{bmatrix} + \begin{bmatrix} -2b & a-3b \\ -2c & b-3c \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -4b & a-3b-2c \\ a-3b-2c & 2b-6c \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$-4b=-1$ $\quad b=0.25$ //

$2b-6c=-1$

$\frac{2(0.25)+1}{6} = c = 0.25$ //

due to symmetry of P we have only $\frac{(n+1)n}{2}$ equations

$$\begin{bmatrix} 0 & -4 & 0 \\ 1 & -3 & -2 \\ 0 & 2 & -6 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

these linear equations are decoupled.

$$\therefore \quad P = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$$

use Sylvester's theorem.

$pm1 = 1.25 > 0$

$pm2 = \begin{vmatrix} 1.25 & 0.25 \\ 0.25 & 0.25 \end{vmatrix} = 0.25 > 0$ $\quad \therefore$ P is the def^{te}

$\therefore$ System is asymptotically stable.

This answer can be verified by solving $|sI-A|=0$, which shows that the system poles are located at -1 and -2 (stable).